Uniform stabilization in weighted Sobolev spaces for the KdV equation posed on the half-line

Ademir F. Pazoto, Lionel Rosier

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February 3, 2010

Abstract

Studied here is the large-time behavior of solutions of the Korteweg-de Vries equation posed on the right half-line under the effect of a localized damping. Assuming as in [20] that the damping is active on a set $(a_0, +\infty)$ with $a_0 > 0$, we establish the exponential decay of the solutions in the weighted spaces $L^2((x+1)^m dx)$ for $m \in \mathbb{N}^*$ and $L^2(e^{2bx}dx)$ for $b > 0$ by a Lyapunov approach. The decay of the spatial derivatives of the solution is also derived.

MSC: Primary: 93D15, 35Q53; Secondary: 93B05.
Key words. Exponential Decay, Korteweg-de Vries equation, Stabilization.

1 Introduction

The Korteweg-de Vries (KdV) equation was first derived as a model for the propagation of small amplitude long water waves along a channel [9, 16, 17]. It has been intensively studied from various aspects for both mathematics and physics since the 1960s when solitons were discovered through solving the KdV equation, and the inverse scattering method, a so-called nonlinear Fourier transform, was invented to seek solitons [14, 22]. It is now well known that the KdV equation is not only a good model for water waves but also a very useful approximation model in nonlinear studies whenever one wishes to include and balance weak nonlinear and dispersive effects.

The initial boundary value problems (IBVP) arise naturally in modeling small-amplitude long waves in a channel with a wavemaker mounted at one end [1, 2, 3, 29]. Such mathematical formulations have received considerable attention in the past, and a satisfactory theory of global well-posedness is available for initial and boundary conditions satisfying physically relevant smoothness and consistency assumptions (see e.g. [1, 4, 6, 7, 11, 12, 13] and the references therein).

The analysis of the long-time behavior of IBVP on the quarter-plane for KdV has also received considerable attention over recent years, and a review of some of the results related to the issues we address here can be found in [5, 7, 19]. For stabilization and controllability issues on the half line, we refer the reader to [20] and [27, 28], respectively.

In this work, we are concerned with the asymptotic behavior of the solutions of the IBVP for the KdV equation posed on the positive half line under the presence of a localized damping represented by the function a; that is,

$$(1)\qquad \begin{cases} u_t + u_x + u_{xxx} + uu_x + a(x)u = 0, & x,\, t \in \mathbb{R}^+, \\ u(0,t) = 0, & t > 0, \\ u(x,0) = u_0(x), & x > 0. \end{cases}$$

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Assuming $a(x) \geq 0$ a.e. and that $u(.,t) \in H^3(\mathbb{R}^+)$, it follows from a simple computation that

$$\frac{dE}{dt} = -\int_0^\infty a(x)|u(x,t)|^2 dx - \frac{1}{2}|u_x(0,t)|^2 \tag{2}$$

where

$$E(t) = \frac{1}{2}\int_0^\infty |u(x,t)|^2 dx \tag{3}$$

is the total energy associated with (1). Then, we see that the term $a(x)u$ plays the role of a feedback damping mechanism and, consequently, it is natural to wonder whether the solutions of (1) tend to zero as $t \to \infty$ and under what rate they decay. When $a(x) > a_0 > 0$ almost everywhere in $\mathbb{R}^+$, it is very simple to prove that $E(t)$ converges to zero as t tends to infinity. The problem of stabilization when the damping is effective only in a subset of the domain is much more subtle. The following result was obtained in [20].

Theorem 1.1 *Assume that the function $a = a(x)$ satisfies the following property*

$$a \in L^\infty(\mathbb{R}^+),\ a \geq 0 \text{ a.e. in } \mathbb{R}^+ \text{ and } a(x) \geq a_0 > 0 \text{ a.e. in } (x_0, +\infty) \tag{4}$$

for some numbers $a_0, x_0 > 0$. Then for all $R > 0$ there exist two numbers $C > 0$ and $\nu > 0$ such that for all $u_0 \in L^2(\mathbb{R}^+)$ with $||u_0||_{L^2(\mathbb{R}^+)} \leq R$, the solution u of (1) satisfies

$$||u(t)||_{L^2(\mathbb{R}^+)} \leq Ce^{-\nu t}||u_0||_{L^2(\mathbb{R}^+)}. \tag{5}$$

Actually, Theorem 1.1 was proved in [20] under the additional hypothesis that

$$a(x) \geq a_0 \text{ a.e. in } (0, \delta) \tag{6}$$

for some $\delta > 0$, but (6) may be dropped by replacing the unique continuation property [20, Lemma 2.4] by [30, Theorem 1.6]. The exponential decay of $E(t)$ is obtained following the methods in [23, 25, 26] which combine multiplier techniques and compactness arguments to reduce the problem to some unique continuation property for weak solutions of KdV.

Along this work we assume that the real-valued function $a = a(x)$ satisfies the condition (4) for some given positive numbers a_0, x_0. In this paper we investigate the stability properties of (1) in the weighted spaces introduced by Kato in [15]. More precisely, for $b > 0$ and $m \in \mathbb{N}$, we prove that the solution u exponentially decays to 0 in L^2_b and $L^2_{(x+1)^m dx}$ (if $u(0)$ belongs to one of these spaces), where

$$L^2_b = \{u : \mathbb{R}^+ \to \mathbb{R};\ \int_0^\infty |u(x)|^2 e^{2bx} dx < \infty\},$$

$$L^2_{(x+1)^m dx} = \{u : \mathbb{R}^+ \to \mathbb{R};\ \int_0^\infty |u(x)|^2 (x+1)^m dx < \infty\}.$$

The following weighted Sobolev spaces

$$H^s_b = \{u : \mathbb{R}^+ \to \mathbb{R};\ \partial^i_x u \in L^2_b \text{ for } 0 \leq i \leq s;\ u(0) = 0 \text{ if } s \geq 1\}$$

and

$$H^s_{(x+1)^m dx} = \{u : \mathbb{R}^+ \to \mathbb{R};\ \partial^i_x u \in L^2_{(x+1)^{m-i} dx} \text{ for } 0 \leq i \leq s;\ u(0) = 0 \text{ if } s \geq 1\},$$

endowed with their usual inner products, will be used thereafter. Note that $H^0_b = L^2_b$ and that $H^0_{(x+1)^m dx} = L^2_{(x+1)^m dx}$.

The exponential decay in $L^2_{(x+1)^m dx}$ is obtained by constructing a convenient Lyapunov function (which actually decreases strictly on the sequence of times $\{kT\}_{k\ge 0}$) by induction on m. For $u_0 \in L^2_{(x+1)^m dx}$, we also prove the following estimate

$$||u(t)||_{H^1_{(x+1)^m dx}} \le C \frac{e^{-\mu t}}{\sqrt{t}} ||u_0||_{L^2_{(x+1)^m dx}} \tag{7}$$

in two situations: (i) $m = 1$ and $||u_0||_{L^2_{(x+1)^m dx}}$ is arbitrarily large; (ii) $m \ge 2$ and $||u_0||_{L^2_{(x+1)^m dx}}$ is small enough. In the situation (ii), we first establish a similar estimate for the linearized system and next apply the contraction mapping principle in a space of functions fulfilling the exponential decay. Note that (7) combines the (global) Kato smoothing effect to the exponential decay.

The exponential decay in L^2_b is established for any initial data $u_0 \in L^2_b$ under the additional assumption that $4b^3 + b < a_0$. Next, we can derive estimates of the form

$$||u(t)||_{H^s_b} \le C \frac{e^{-\mu t}}{t^{s/2}} ||u_0||_{L^2_b}$$

for any $s \ge 1$, revealing that $u(t)$ decays exponentially to 0 in strong norms.

It would be interesting to see if such results are still true when the function a has a smaller support. It seems reasonable to conjecture that similar positive results can be derived when the support of a contains a set of the form $\cup_{k\ge 1}[ka_0, ka_0 + b_0]$ where $0 < b_0 < a_0$, while a negative result probably holds when the support of a is a finite interval, as the L^2 norm of a soliton-like initial data may not be sufficiently dissipated over time. Such issues will be discussed elsewhere.

The plan of this paper is as follows. Section 2 is devoted to global well-posedness results in the weighted spaces L^2_b and $L^2_{(x+1)^2 dx}$. In section 3, we prove the exponential decay in $L^2_{(x+1)^m dx}$ and L^2_b, and establish the exponential decay of the derivatives as well.

2 Global well-posedness

2.1 Global well-posedness in $\mathbf{L^2_b}$

Fix any $b > 0$. To begin with, we apply the classical semigroup theory to the linearized system

$$\begin{cases} u_t + u_x + u_{xxx} + a(x)u = 0, \quad x,\, t \in \mathbb{R}^+, \\ u(0,t) = 0, \quad t > 0, \\ u(x,0) = u_0(x), \quad x > 0. \end{cases} \tag{8}$$

Let us consider the operator

$$A : D(A) \subset L^2_b \to L^2_b$$

with domain

$$D(A) = \{u \in L^2_b;\ \partial_x^i u \in L^2_b \text{ for } 1 \le i \le 3 \text{ and } u(0) = 0\}$$

defined by

$$Au = -u_{xxx} - u_x - a(x)u.$$

Then, the following result holds.

Lemma 2.1 *The operator A defined above generates a continuous semigroup of operators $(S(t))_{t\ge 0}$ in L^2_b.*

Proof. We first introduce the new variable $v = e^{bx}u$ and consider the following (IBVP)

$$
\begin{cases} v_t + (\partial_x - b)v + (\partial_x - b)^3 v + a(x)v = 0, & x,\, t \in \mathbb{R}^+, \\ v(0,t) = 0, & t > 0, \\ v(x,0) = v_0(x) = e^{bx}u_0(x), & x > 0. \end{cases} \tag{9}
$$

Clearly, the operator $B : D(B) \subset L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$ with domain

$$D(B) = \{u \in H^3(\mathbb{R}^+);\ u(0) = 0\}$$

defined by

$$Bv = -(\partial_x - b)v - (\partial_x - b)^3 v - a(x)v$$

is densely defined and closed. So, we are done if we prove that for some real number λ the operator $B - \lambda$ and its adjoint $B^* - \lambda$ are both dissipative in $L^2(\mathbb{R}^+)$. It is readily seen that $B^* : D(B^*) \subset L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$ is given by $B^*v = (\partial_x + b)v + (\partial_x + b)^3 v - a(x)v$ with domain

$$D(B^*) = \{v \in H^3(\mathbb{R}^+);\ v(0) = v'(0) = 0\}.$$

Pick any $v \in D(B)$. After some integration by parts, we obtain that

$$(Bv, v)_{L^2} = -\frac{1}{2}v_x^2(0) - 3b\int_0^\infty v_x^2 dx + (b + b^3)\int_0^\infty v^2 dx - \int_0^\infty a(x)v^2 dx,$$

that is,

$$([B - (b^3 + b)]v, v)_{L^2} \le 0.$$

Analogously, we deduce that for any $v \in D(B^*)$

$$(v, [B^* - (b^3 + b)]v)_{L^2} \le 0$$

which completes the proof. ■

The following linear estimates will be needed.

Lemma 2.2 *Let $u_0 \in L_b^2$ and $u = S(\cdot)u_0$. Then, for any $T > 0$*

$$\frac{1}{2}\int_0^\infty |u(x,T)|^2 dx - \frac{1}{2}\int_0^\infty |u_0(x)|^2 dx + \int_0^T\int_0^\infty a(x)|u|^2 dx dt + \frac{1}{2}\int_0^T u_x^2(0,t)dt = 0 \tag{10}$$

$$
\begin{aligned}
&\frac{1}{2}\int_0^\infty |u(x,T)|^2 e^{2bx} dx - \frac{1}{2}\int_0^\infty |u_0(x)|^2 e^{2bx} dx + 3b\int_0^T\int_0^\infty u_x^2 e^{2bx} dx dt \\
&-(4b^3 + b)\int_0^T\int_0^\infty u^2 e^{2bx} dx dt + \int_0^T\int_0^\infty a(x)|u|^2 e^{2bx} dx dt + \frac{1}{2}\int_0^T u_x^2(0,t)dt = 0.
\end{aligned} \tag{11}
$$

As a consequence,

$$||u||_{L^\infty(0,T;L_b^2)} + ||u_x||_{L^2(0,T;L_b^2)} \le C\,||u_0||_{L_b^2}, \tag{12}$$

where $C = C(T)$ is a positive constant.

Proof. Pick any $u_0 \in D(A)$. Multiplying the equation in (1) by u and integrating over $(0,+\infty) \times (0,T)$, we obtain (10). Then, the identity may be extended to any initial state $u_0 \in L^2_b$ by a density argument. To derive (11) we first multiply the equation by $(e^{2bx}-1)u$ and integrate by parts over $(0,+\infty) \times (0,T)$ to deduce that

$$\frac{1}{2}\int_0^\infty |u(x,T)|^2(e^{2bx}-1)dx - \frac{1}{2}\int_0^\infty |u_0(x)|^2(e^{2bx}-1)dx +$$
$$+3b\int_0^T\int_0^\infty u_x^2 e^{2bx}dxdt - (4b^3+b)\int_0^T\int_0^\infty u^2e^{2bx}dxdt +$$
$$+\int_0^T\int_0^\infty a(x)|u|^2(e^{2bx}-1)dxdt = 0.$$

Adding the above equality and (10) hand to hand, we obtain (11) using the same density argument. Then, Gronwall inequality, (4) and (11) imply that

$$||u||_{L^\infty(0,T;L^2_b)} \le C\,||u_0||_{L^2_b},$$

with $C = C(T) > 0$. This estimate together with (11) gives us

$$||u_x||_{L^2(0,T;L^2_b)} \le C\,||u_0||_{L^2_b},$$

where $C = C(T)$ is a positive constant. ■

The global well-posedness result reads as follows:

Theorem 2.3 *For any $u_0 \in L^2_b$ and any $T > 0$, there exists a unique solution $u \in C([0,T];L^2_b) \cap L^2(0,T;H^1_b)$ of (1).*

Proof. By computations similar to those performed in the proof of Lemma 2.2, we obtain that for any $f \in C^1([0,T];L^2_b)$ and any $u_0 \in D(A)$, the solution u of the system

$$\begin{cases} u_t + u_x + u_{xxx} + a(x)u = f, & x \in \mathbb{R}^+,\ t \in (0,T), \\ u(0,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in \mathbb{R}^+, \end{cases}$$

fulfills

$$\sup_{0\le t\le T} ||u(t)||_{L^2_b} + (\int_0^T\int_0^\infty |u_x|^2 e^{2bx}dxdt)^{\frac{1}{2}} \le C\left(||u_0||_{L^2_b} + \int_0^T ||f||_{L^2_b}dt\right) \tag{13}$$

for some constant $C = C(T)$ nondecreasing in T. A density argument yields that $u \in C([0,T];L^2_b)$ when $f \in L^1(0,T;L^2_b)$ and $u_0 \in L^2_b$.

Let $u_0 \in L^2_b$ be given. To prove the existence of a solution of (1) we introduce the map Γ defined by

$$(\Gamma u)(t) = S(t)u_0 + \int_0^t S(t-s)N(u(s))\,ds$$

where $N(u) = -uu_x$, and the space

$$F = C([0,T];L^2_b) \cap L^2(0,T;H^1_b)$$

endowed with its natural norm. We shall prove that Γ has a fixed-point in some ball $B_R(0)$ of F. We need the following

CLAIM 1. If $u \in H_b^1$ then

$$||u^2 e^{2bx}||_{L^\infty(\mathbb{R}^+)} \leq (2+2b)\, ||u||_{L_b^2} ||u||_{H_b^1}.$$

From Cauchy-Schwarz inequality, we get for any $\overline{x} \in \mathbb{R}^+$

$$\begin{aligned} u^2(\overline{x})e^{2b\overline{x}} &= \int_0^{\overline{x}} [u^2 e^{2bx}]_x dx = \int_0^{\overline{x}} [2uu_x e^{2bx} + 2bu^2 e^{2bx}]dx \\ &\leq 2(\int_0^\infty u^2 e^{2bx}dx)^{\frac{1}{2}} (\int_0^\infty u_x^2 e^{2bx} dx)^{\frac{1}{2}} + 2b\int_0^\infty u^2 e^{2bx}dx \leq (2+2b)||u||_{L_b^2}||u||_{H_b^1} \end{aligned}$$

which guarantees that Claim 1 holds.

CLAIM 2. There exists a constant $K > 0$ such that for $0 < T \leq 1$

$$||\Gamma(u) - \Gamma(v)||_F \leq KT^{\frac{1}{4}}(||u||_F + ||v||_F)||u-v||_F, \quad \forall\, u, v\, \in F.$$

According to the previous analysis,

$$||\Gamma(u) - \Gamma(v)||_F \leq C||uu_x - vv_x||_{L^1(0,T;L_b^2)}.$$

So, applying triangular inequality and Hölder inequality, we have

$$\begin{aligned} ||\Gamma(u) - \Gamma(v)||_F \leq C\{&||u-v||_{L^2(0,T;L^\infty(0,\infty))}||u||_{L^2(0,T;H_b^1)} + \\ &+||v||_{L^2(0,T;L^\infty(0,\infty))}||u-v||_{L^2(0,T;H_b^1)}\}. \end{aligned} \tag{14}$$

Now, by Claim 1, we have

$$||u||_{L^2(0,T;L^\infty(0,\infty))} \leq C\,T^{\frac{1}{4}}||u||^{\frac{1}{2}}_{L^\infty(0,T;L_b^2)}||u||^{\frac{1}{2}}_{L^2(0,T;H_b^1)}. \tag{15}$$

Then, combining (14) and (15), we deduce that

$$||\Gamma(u) - \Gamma(v)||_F \leq C\,T^{\frac{1}{4}}\{\, ||u||_F + ||v||_F\,\}||u-v||_F. \tag{16}$$

Let $T > 0$, $R > 0$ be numbers whose values will be specified later, and let $u \in B_R(0) \subset F$ be given. Then, by Claim 2 and Lemma 2.2, $\Gamma u \in F$ and

$$||\Gamma u||_F \leq C\,(\,||u_0||_{L_b^2} + T^{\frac{1}{4}}||u||_F^2\,).$$

Consequently, for $R = 2C||u_0||_{L_b^2}$ and $T > 0$ small enough, Γ maps $B_R(0)$ into itself. Moreover, we infer from (16) that this mapping contracts if T is small enough. Then, by the contraction mapping theorem, there exists a unique solution $u \in B_R(0) \subset F$ to the problem (1) for T small enough.

In order to prove that this solution is global, we need some a priori estimates. So, we proceed as in the proof of Lemma 2.2 to obtain for the solution u of (1)

$$\frac{1}{2}\int_0^\infty |u(x,T)|^2 dx - \frac{1}{2}\int_0^\infty |u_0(x)|^2 dx + \int_0^T\int_0^\infty a(x)|u|^2 dxdt + \frac{1}{2}\int_0^T u_x^2(0,t)dt = 0 \tag{17}$$

and

$$\begin{aligned} \frac{1}{2}\int_0^\infty |u(x,T)|^2 e^{2bx}dx - \frac{1}{2}\int_0^\infty |u_0(x)|^2 e^{2bx}dx + \frac{1}{2}\int_0^T u_x^2(0,t)dt \\ + 3b\int_0^T\int_0^\infty u_x^2 e^{2bx}dxdt - (4b^3+b)\int_0^T\int_0^\infty u^2 e^{2bx}dxdt \\ + \int_0^T\int_0^\infty a(x)|u|^2 e^{2bx}dxdt - \frac{2b}{3}\int_0^T\int_0^\infty u^3 e^{2bx}dxdt = 0. \end{aligned} \tag{18}$$

First, observe that

$$|\int_0^\infty u^2 e^{2bx}dx| = |-\frac{1}{b}\int_0^\infty uu_x e^{2bx}dx| \leq \frac{1}{b}(\int_0^\infty u^2 e^{2bx}dx)^{\frac{1}{2}}(\int_0^\infty u_x^2 e^{2bx}dx)^{\frac{1}{2}},$$

therefore,

$$\int_0^\infty u^2 e^{2bx}dx \leq \frac{1}{b^2}\int_0^\infty u_x^2 e^{2bx}dx.$$

Combined to Claim 1, this yields

$$||u(x)e^{bx}||_{L^\infty(\mathbb{R}^+)} \leq C||u_x||_{L_b^2}.$$

On the other hand, it follows from (17) that

$$||u(t)||_{L^2(\mathbb{R}^+)} \leq ||u_0||_{L^2(\mathbb{R}^+)},$$

hence

$$\begin{aligned}\int_0^T\int_0^\infty |u|^3 e^{2bx}dxdt &\leq \int_0^T ||ue^{bx}||_{L^\infty(\mathbb{R}^+)}(\int_0^\infty |u|^2 e^{bx}dx)dt \\ &\leq C\int_0^T ||u_x||_{L_b^2}||u||_{L_b^2}||u||_{L^2}dt \\ &\leq \delta||u_x||^2_{L^2(0,T;L_b^2)} + C_\delta||u||^2_{L^2(0,T;L_b^2)},\end{aligned}$$

where $\delta > 0$ is arbitrarily chosen and $C = C(b, \delta, ||u_0||_{L^2(\mathbb{R}^+)})$ is a positive constant. Combining this inequality (with $\delta < 9/2$) to (18) results in

$$||u(T)||^2_{L_b^2} \leq ||u_0||^2_{L_b^2} + C\int_0^T ||u||^2_{L_b^2}dt$$

where $C = C(b, ||u_0||_{L^2(\mathbb{R}^+)})$ does not depend on T. It follows from Gronwall lemma that

$$||u(T)||^2_{L_b^2} \leq ||u_0||^2_{L_b^2}e^{CT}$$

for all $T > 0$, which gives the global well-posedness. ■

2.2 Global well-posedness in $L^2_{(x+1)^2dx}$

Definition 2.4 *For $u_0 \in L^2_{(x+1)^2dx}$ and $T > 0$, we denote by a* mild solution *of* (1) *any function $u \in C([0,T]; L^2_{(x+1)^2dx}) \cap L^2(0,T; H^1_{(x+1)^2dx})$ which solves* (1)*, and such that for some $b > 0$ and some sequence $\{u_{n,0}\} \subset L_b^2$ we have*

$$\begin{aligned}&u_{n,0} \to u_0 \quad \textit{strongly in} \quad L^2_{(x+1)^2dx}, \\ &u_n \to u \textit{ weakly}* \textit{ in } L^\infty(0,T; L^2_{(x+1)^2dx}), \\ &u_n \to u \textit{ weakly in } L^2(0,T; H^1_{(x+1)^2dx}),\end{aligned}$$

u_n denoting the solution of (1) *emanating from $u_{n,0}$ at $t = 0$.*

Theorem 2.5 *For any $u_0 \in L^2_{(x+1)^2dx}$ and any $T > 0$, there exists a unique mild solution $u \in C([0,T]; L^2_{(x+1)^2dx}) \cap L^2(0,T; H^1_{(x+1)^2dx})$ of* (1).

Proof. We prove the existence and the uniqueness in two steps.

Step 1. Existence

Since the embedding $L^2_b \subset L^2_{(x+1)^2dx}$ is dense, for any given $u_0 \in L^2_{(x+1)^2dx}$ we may construct a sequence $\{u_{n,0}\} \subset L^2_b$ such that $u_{n,0} \to u_0$ in $L^2_{(x+1)^2dx}$ as $n \to \infty$. For each n, let u_n denote the solution of (1) emanating from $u_{n,0}$ at $t = 0$, which is given by Theorem 2.3. Then $u_n \in C([0,T]; L^2_b) \cap L^2(0,T; H^1_b)$ and it solves

$$u_{n,t} + u_{n,x} + u_{n,xxx} + u_n u_{n,x} + a(x)u_n = 0, \tag{19}$$

$$u_n(0,t) = 0 \tag{20}$$

$$u_n(x,0) = u_{n,0}(x). \tag{21}$$

Multiplying (19) by $(x+1)^2 u_n$ and integrating by parts, we obtain

$$\begin{aligned} &\frac{1}{2}\int_0^\infty (x+1)^2 |u_n(x,T)|^2 dx + 3\int_0^T\int_0^\infty (x+1)|u_{n,x}|^2 dxdt + \frac{1}{2}\int_0^T |u_{n,x}(0,t)|^2 dt \\ &- \int_0^T\int_0^\infty (x+1)|u_n|^2 dxdt - \frac{2}{3}\int_0^T\int_0^\infty (x+1)u_n^3\, dxdt + \int_0^T\int_0^\infty (x+1)^2 u_n^2 a(x) dx \\ &\quad = \frac{1}{2}\int_0^\infty (x+1)^2 |u_{n,0}(x)|^2 dx. \end{aligned} \tag{22}$$

Scaling in (19) by u_n gives

$$\begin{aligned} &\frac{1}{2}\int_0^\infty |u_n(x,T)|^2 dx + \frac{1}{2}\int_0^T |u_{n,x}(0,t)|^2 dt + \int_0^T\int_0^\infty a(x)|u_n(x,t)|^2 dxdt \\ &= \frac{1}{2}\int_0^\infty |u_{n,0}(x)|^2 dx, \end{aligned}$$

hence

$$||u_n||_{L^2(\mathbb{R}^+)} \le ||u_{n,0}||_{L^2(\mathbb{R}^+)} \le C \tag{23}$$

where $C = C(||u_0||_{L^2(\mathbb{R}^+)})$. It follows that

$$\begin{aligned} \frac{2}{3}\int_0^\infty (x+1)|u_n|^3 dx &\le \frac{2\sqrt{2}}{3} ||u_{n,x}||^{\frac{1}{2}}_{L^2(\mathbb{R}^+)} ||u_n||^{\frac{3}{2}}_{L^2(\mathbb{R}^+)} ||(x+1)u_n||_{L^2(\mathbb{R}^+)} \\ &\le \int_0^\infty (x+1)|u_{n,x}|^2 dx + C\int_0^\infty (x+1)^2 |u_n|^2 dx \end{aligned} \tag{24}$$

which, combined to (22), gives

$$\begin{aligned} &\frac{1}{2}\int_0^\infty (x+1)^2 |u_n(x,T)|^2 dx + 2\int_0^T\int_0^\infty (x+1)|u_{n,x}|^2 dxdt + \frac{1}{2}\int_0^T |u_{n,x}(0,t)|^2 dt \\ &\quad \le \frac{1}{2}\int_0^\infty (x+1)^2 |u_{n,0}(x)|^2 dx + C\int_0^T\int_0^\infty (x+1)^2 |u_n(x,t)|^2 dxdt. \end{aligned}$$

An application of Gronwall's lemma yields

$$\begin{aligned} ||u_n||_{L^\infty(0,T;L^2_{(x+1)^2dx})} &\le C(T, ||u_{n,0}||_{L^2_{(x+1)^2dx}}), \\ ||u_{n,x}||_{L^2(0,T;H^1_{(x+1)^2dx})} &\le C(T, ||u_{n,0}||_{L^2_{(x+1)^2dx}}), \\ ||u_{n,x}(0,.)||_{L^2(0,T)} &\le C(T, ||u_{n,0}||_{L^2_{(x+1)^2dx}}). \end{aligned}$$

Therefore, there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that

$$\begin{cases} u_n \rightharpoonup u \text{ weakly } * \text{ in } L^\infty(0,T;L^2_{(x+1)^2dx}), \\ u_n \rightharpoonup u \text{ weakly in } L^2(0,T;H^1_{(x+1)^2dx}), \\ u_{n,x}(0,.) \rightharpoonup u_x(0,.) \text{ weakly in } L^2(0,T). \end{cases}$$

Note that, for all $L>0$, $\{u_n\}$ is bounded in $L^2(0,T;H^1(0,L))\cap H^1(0,T;H^{-2}(0,L))$, hence by Aubin's lemma, we have (after extracting a subsequence if needed)

$$u_n \to u \ \text{ strongly in } \ L^2(0,T;L^2(0,L)) \text{ for all } L>0.$$

This gives that $u_nu_{n,x} \to uu_x$ in the sense of distributions, hence the limit $u\in L^\infty(0,T;L^2_{(x+1)^2dx})\cap L^2(0,T;H^1_{(x+1)^2dx})$ is a solution of (1). Let us check that $u\in C([0,T];L^2_{(x+1)^2dx})$. Since $u\in C([0,T];H^{-2}(\mathbb{R}^+))\cap L^\infty(0,T;L^2_{(x+1)^2dx})$, we have that $u\in C_w([0,T];L^2_{(x+1)^2dx})$ (see e.g. [21]), where $C_w([0,T];L^2_{(x+1)^2dx})$ denotes the space of sequentially weakly continuous functions from $[0,T]$ into $L^2_{(x+1)^2dx}$.

We claim that $u\in L^3(0,T;L^3(\mathbb{R}^+))$. Indeed, from Moser estimate (see [31])

$$||u||_{L^\infty(\mathbb{R}^+)} \le \sqrt{2}||u_x||^{\frac{1}{2}}_{L^2(\mathbb{R}^+)}||u||^{\frac{1}{2}}_{L^2(\mathbb{R}^+)} \tag{25}$$

and Young inequality we get

$$\int_0^\infty |u|^3dx \le ||u||_{L^\infty}||u||^2_{L^2} \le \sqrt{2}||u_x||^{\frac{1}{2}}_{L^2}||u||^{\frac{5}{2}}_{L^2} \le \varepsilon||u_x||^2_{L^2} + c_\varepsilon||u||^{\frac{10}{3}}_{L^2} \tag{26}$$

where $\varepsilon>0$ is arbitrarily chosen and c_ε denotes some positive constant. Since $u\in C_w([0,T];L^2_{(x+1)^2dx})\cap L^2(0,T;H^1_{(x+1)^2dx})$, it follows that $u\in L^3(0,T;L^3(\mathbb{R}^+))$. On the other hand, $u(0,t)=0$ for $t\in(0,T)$ and $u_x(0,.)\in L^2(0,T)$. Scaling in (1) by $(x+1)^2u$ yields for all $t_1,t_2\in(0,T)$

$$\begin{aligned} &\frac{1}{2}\int_0^\infty (x+1)^2|u(x,t_2)|^2dx - \frac{1}{2}\int_0^\infty (x+1)^2|u(x,t_1)|^2dx \\ &= -3\int_{t_1}^{t_2}\int_0^\infty (x+1)|u_x|^2dxdt - \frac{1}{2}\int_{t_1}^{t_2}|u_x(0,t)|^2dt + \int_{t_1}^{t_2}\int_0^\infty (x+1)|u|^2dxdt \\ &+\frac{2}{3}\int_{t_1}^{t_2}\int_0^\infty (x+1)u^3dxdt - \int_{t_1}^{t_2}\int_0^\infty (x+1)^2a(x)|u|^2dxdt. \end{aligned} \tag{27}$$

Therefore $\lim_{t_1\to t_2}\left| ||u(t_2)||^2_{L^2_{(x+1)^2dx}} - ||u(t_1)||^2_{L^2_{(x+1)^2dx}}\right| = 0$. Combined to the fact that $u\in C_w([0,T];L^2_{(x+1)^2dx})$ this yields $u\in C([0,T],L^2_{(x+1)^2dx})$.

Step 2. Uniqueness

Here, C will denote a universal constant which may vary from line to line. Pick $u_0\in L^2_{(x+1)^2dx}$, and let $u,v\in C([0,T];L^2_{(x+1)^2dx})\cap L^2(0,T;H^1_{(x+1)^2dx})$ be two mild solutions of (1). Pick two sequences $\{u_{n,0}\}$, $\{v_{n,0}\}$ in L^2_b for some $b>0$ such that

$$u_{n,0} \to u_0 \ \text{ strongly in } \ L^2_{(x+1)^2dx}, \tag{28}$$

$$u_n \to u \text{ weakly} * \text{ in } L^\infty(0,T;L^2_{(x+1)^2dx}), \tag{29}$$

$$u_n \to u \text{ weakly in } L^2(0,T;H^1_{(x+1)^2dx}) \tag{30}$$

and also

$$v_{n,0} \to u_0 \ \text{ strongly in } \ L^2_{(x+1)^2dx}, \tag{31}$$

$$v_n \to v \text{ weakly} * \text{ in } L^\infty(0,T;L^2_{(x+1)^2dx}), \tag{32}$$

$$v_n \to v \text{ weakly in } L^2(0,T;H^1_{(x+1)^2dx}). \tag{33}$$

We shall prove that $w = u - v$ vanishes on $\mathbb{R}^+ \times [0,T]$ by providing some estimate for $w_n = u_n - v_n$. Note first that w_n solves the system

$$w_{n,t} + w_{n,x} + w_{n,xxx} + aw_n = f_n = v_n v_{n,x} - u_n u_{n,x}, \tag{34}$$

$$w_n(0,t) = 0, \tag{35}$$

$$w_n(x,0) = w_{n,0}(x) = u_{n,0}(x) - v_{n,0}(x). \tag{36}$$

Scaling in (34) by $(x+1)w_n$ yields

$$\begin{aligned}
&\frac{1}{2}\int_0^\infty (x+1)|w_n(x,t)|^2dx + \frac{3}{2}\int_0^t\int_0^\infty |w_{n,x}|^2dxd\tau - \frac{1}{2}\int_0^t\int_0^\infty |w_n|^2dxd\tau \\
&\le \frac{1}{2}\int_0^\infty (x+1)|w_{n,0}|^2dx + \int_0^t (\int_0^\infty (x+1)|w_n|^2dx)^{\frac{1}{2}}(\int_0^\infty (x+1)|f_n|^2dx)^{\frac{1}{2}}d\tau \\
&\le \frac{1}{2}\int_0^\infty (x+1)|w_{n,0}|^2dx + \frac{1}{4}\sup_{0<\tau<t}\int_0^\infty (x+1)|w_n(x,\tau)|^2dx \\
&\quad + [\int_0^T (\int_0^\infty (x+1)|f_n|^2dx)^{\frac{1}{2}}d\tau]^2.
\end{aligned}$$

Since $||w_n(t)||_{L^2(\mathbb{R}^+)} \le ||w_n(t)||_{L^2_{(x+1)dx}}$, this yields for $T < 1/10$

$$\begin{aligned}
&\sup_{0<t<T}\int_0^\infty (x+1)|w_n(x,t)|^2dx + \int_0^T\int_0^\infty |w_{n,x}|^2dxdt \\
&\le C[\int_0^\infty (x+1)|w_{n,0}(x)|^2dx + \left(\int_0^T (\int_0^\infty (x+1)|f_n|^2dx)^{\frac{1}{2}}d\tau\right)^2].
\end{aligned} \tag{37}$$

It remains to estimate $\int_0^T(\int_0^\infty (x+1)|f_n|^2dx)^{\frac{1}{2}}dt$. We split f_n into

$$f_n = (v_n - u_n)v_{n,x} + u_n(v_{n,x} - u_{n,x}) = f_n^1 + f_n^2.$$

We have that

$$\begin{aligned}
\int_0^T (\int_0^\infty (x+1)|f_n^1|^2dx)^{\frac{1}{2}}dt &= \int_0^T (\int_0^\infty (x+1)|w_n|^2|v_{n,x}|^2dx)^{\frac{1}{2}}dt \\
&\le \int_0^T ||w_n||_{L^\infty(\mathbb{R}^+)}(\int_0^\infty (x+1)|v_{n,x}|^2dx)^{\frac{1}{2}}dt \\
&\le (\int_0^T ||w_n||^2_{L^\infty(\mathbb{R}^+)}dt)^{\frac{1}{2}}(\int_0^T\int_0^\infty (x+1)|v_{n,x}|^2dxdt)^{\frac{1}{2}}.
\end{aligned}$$

By Sobolev embedding, we have that

$$\begin{aligned}
(\int_0^T ||w_n||^2_{L^\infty(\mathbb{R}^+)}dt)^{\frac{1}{2}} &\le (\int_0^T ||w_n||^2_{H^1(\mathbb{R}^+)}dt)^{\frac{1}{2}} \\
&\le \sqrt{T}\sup_{0<t<T}||w_n||_{L^2(\mathbb{R}^+)} + ||w_{n,x}||_{L^2(0,T;L^2(\mathbb{R}^+))}.
\end{aligned}$$

Thus

$$\int_0^T (\int_0^\infty (x+1)|f_n^1|^2 dx)^{\frac{1}{2}} dt \le ||v_{n,x}||_{L^2(0,T;L^2_{(x+1)dx})} \big(\sqrt{T} \sup_{0<t<T} ||w_n||_{L^2(\mathbb{R}^+)} + ||w_{n,x}||_{L^2(0,T;L^2(\mathbb{R}^+))}\big) \tag{38}$$

On the other hand, we have that

$$\begin{aligned}
&\int_0^T (\int_0^\infty (x+1)|f_n^2|^2 dx)^{\frac{1}{2}} dt \\
&= \int_0^T (\int_0^\infty (x+1)|u_n|^2 |w_{n,x}|^2 dx)^{\frac{1}{2}} dt \\
&\le \int_0^T ||(x+1)^{\frac{1}{2}} u_n||_{L^\infty(\mathbb{R}^+)} ||w_{n,x}||_{L^2(\mathbb{R}^+)} dt \\
&\le C \int_0^T \big(||(x+1)^{\frac{1}{2}} u_n||_{L^2(\mathbb{R}^+)} + ||(x+1)^{\frac{1}{2}} u_{n,x}||_{L^2(\mathbb{R}^+)}\big) ||w_{n,x}||_{L^2(\mathbb{R}^+)} dt \\
&\le C \Big(\sqrt{T} ||(x+1) u_n||_{L^\infty(0,T;L^2(\mathbb{R}^+))} \\
&\qquad + ||(x+1)^{\frac{1}{2}} u_{n,x}||_{L^2(0,T,L^2(\mathbb{R}^+))} \Big) ||w_{n,x}||_{L^2(0,T;L^2(\mathbb{R}^+))}.
\end{aligned} \tag{39}$$

Gathering together (37), (38) and (39), we conclude that for $T < 1/10$

$$h_n(T) \le K_n(T) h_n(T) + C ||w_{n,0}||^2_{L^2_{(x+1)dx}}$$

where

$$h_n(t) := \sup_{0<\tau<T} \int_0^\infty (x+1)|w_n(x,\tau)|^2 dx + \int_0^T \int_0^\infty |w_{n,x}|^2 dx dt \tag{40}$$

$$\begin{aligned}
K_n(T) \le\ & C \Big(\int_0^T \int_0^\infty (x+1)|v_{n,x}|^2 dx dt + T ||(x+1) u_n||^2_{L^\infty(0,T;L^2(\mathbb{R}^+))} \\
& + \int_0^T \int_0^\infty (x+1)|u_{n,x}|^2 dx dt \Big)
\end{aligned} \tag{41}$$

and C denotes a universal constant. The following claim is needed.
Claim 3.

$$\lim_{T\to 0} \limsup_{n\to\infty} \int_0^T \int_0^\infty (x+1)|u_{n,x}|^2 dx dt = 0, \quad \lim_{T\to 0} \limsup_{n\to\infty} \int_0^T \int_0^\infty (x+1)|v_{n,x}|^2 dx dt = 0.$$

Clearly, it is sufficient to prove the claim for the sequence $\{u_n\}$ only. From (27) applied with $u = u_n$ on $[0,T]$, we obtain

$$\begin{aligned}
&\frac{1}{2} \int_0^\infty (x+1)^2 |u_n(x,T)|^2 dx + 3 \int_0^T \int_0^\infty (x+1)|u_{n,x}|^2 dx dt \\
&\le \frac{1}{2} \int_0^\infty (x+1)^2 |u_{n,0}|^2 dx + \int_0^T \int_0^\infty (x+1)|u_n|^2 dx dt + \frac{2}{3} \int_0^T \int_0^\infty (x+1)|u_n|^3 dx dt.
\end{aligned}$$

Combined to (23)-(24), this gives

$$\begin{aligned}
&||u_n(T)||^2_{L^2_{(x+1)^2dx}} + \int_0^T \int_0^\infty (x+1)|u_{n,x}|^2 dx dt \\
&\le ||u_{n,0}||^2_{L^2_{(x+1)^2dx}} + C \int_0^T ||u_n||^2_{L^2_{(x+1)^2dx}} dt.
\end{aligned} \tag{42}$$

It follows from Gronwall lemma that

$$||u_n(t)||^2_{L^2_{(x+1)^2dx}} \le ||u_{n,0}||^2_{L^2_{(x+1)^2dx}} e^{Ct} \tag{43}$$

Using (43) in (42) and taking the limit sup as $n \to \infty$ gives for a.e. T

$$||u(T)||^2_{L^2_{(x+1)^2dx}} + \limsup_{n\to\infty} \int_0^T \int_0^\infty |u_{n,x}|^2 dxdt \le e^{CT} ||u_0||^2_{L^2_{(x+1)^2dx}}$$

As u is continuous from $\mathbb{R}^+$ to $L^2_{(x+1)^2dx}$, we infer that

$$\lim_{T\to 0} \limsup_{n\to\infty} \int_0^T \int_0^\infty |u_{n,x}|^2 dxdt = 0.$$

The claim is proved. Therefore, we have that for $T > 0$ small enough and n large enough, $K_n(T) < \frac{1}{2}$, and hence

$$h_n(T) \le 2C ||w_n(0)||^2_{L^2_{(x+1)dx}}.$$

This yields

$$||u - v||^2_{L^\infty(0,T;L^2_{(x+1)dx})} \le \liminf_{n\to\infty} h_n(T) \le 2C \liminf_{n\to\infty} ||w_n(0)||^2_{L^2_{(x+1)dx}} = 0$$

and $u = v$ for $0 < t < T$. This proves the uniqueness for T small enough. The general case follows by a classical argument. ■

Remark 2.6 *1. If we assume only that $u_0 \in L^2_{(x+1)dx}$, then a proof similar to Step 1 gives the existence of a mild solution $u \in C([0,T]; L^2_{(x+1)dx}) \cap L^2(0,T; H^1_{(x+1)dx})$ of (1). The uniqueness of such a solution is open. The existence and uniqueness of a solution issuing from $u_0 \in L^2_{(x+1)dx}$ in a class of functions involving a Bourgain norm has been given in [13].*

2. If $u_0 \in L^2_{(x+1)^m dx}$ with $m \ge 3$, then $u \in C([0,T]; L^2_{(x+1)^m dx}) \cap L^2(0,T; H^1_{(x+1)^m dx})$ for all $T > 0$ (see below Theorem 3.1).

3 Asymptotic Behavior

3.1 Decay in $L^2_{(x+1)^m dx}$

Theorem 3.1 *Assume that the function $a = a(x)$ satisfies (4). Then, for all $R > 0$ and $m \ge 1$, there exist numbers $C > 0$ and $\nu > 0$ such that*

$$||u(t)||_{L^2_{(x+1)^m dx}} \le C\, e^{-\nu t} ||u_0||_{L^2_{(x+1)^m dx}}$$

for any solution given by Theorem 2.5, whenever $||u_0||_{L^2_{(x+1)^m dx}} \le R$.

Proof. The proof will be done by induction in m. We set

$$V_0(u) = E(u) = \frac{1}{2} \int_0^\infty u^2 dx \tag{44}$$

and define the Lyapunov function V_m for $m \ge 1$ in an inductive way

$$V_m(u) = \frac{1}{2} \int_0^\infty (x+1)^m u^2 dx + d_{m-1} V_{m-1}(u), \tag{45}$$

where $d_{m-1} > 0$ is chosen sufficiently large (see below).

Suppose first that $m = 1$ and put $V = V_1$. Multiplying the first equation in (1) by u and integrating by parts over $\mathbb{R}^+ \times (0,T)$, we obtain

$$\frac{1}{2}\int_0^\infty |u(x,T)|^2 dx = \frac{1}{2}\int_0^\infty |u_0(x)|^2 dx - \int_0^T\int_0^\infty a(x)|u|^2 dxdt - \frac{1}{2}\int_0^T u_x^2(0,t)dt. \tag{46}$$

Now, multiplying the equation by xu, we deduce that

$$\begin{aligned}&\frac{1}{2}\int_0^\infty x|u(x,T)|^2 dx - \frac{1}{2}\int_0^\infty x|u_0(x)|^2 dx + \frac{3}{2}\int_0^T\int_0^\infty u_x^2 dxdt\\ &-\frac{1}{2}\int_0^T\int_0^\infty u^2 dxdt - \frac{1}{3}\int_0^T\int_0^\infty u^3 dxdt + \int_0^T\int_0^\infty xa(x)|u|^2 dxdt = 0.\end{aligned} \tag{47}$$

Combining (46) and (47) it follows that

$$\begin{aligned}&V(u) - V(u_0) + (d_0+1)\left(\frac{1}{2}\int_0^T u_x^2(0,t)dt + \int_0^T\int_0^\infty a(x)|u|^2 dxdt\right)\\ &+\frac{3}{2}\int_0^T\int_0^\infty u_x^2 dxdt - \frac{1}{2}\int_0^T\int_0^\infty u^2 dxdt - \frac{1}{3}\int_0^T\int_0^\infty u^3 dxdt\\ &\qquad + \int_0^T\int_0^\infty xa(x)|u|^2 dxdt = 0.\end{aligned} \tag{48}$$

The next step is devoted to estimate the nonlinear term in the left hand side of (48). To do that, we first assume that $||u_0||_{L^2} \leq 1$.

By (26) we have that

$$\int_0^\infty |u|^3 dx \leq \varepsilon ||u_x||_{L^2}^2 + c_\varepsilon ||u||_{L^2}^{\frac{10}{3}}$$

for any $\varepsilon > 0$ and some constant $c_\varepsilon > 0$. Thus, if $||u_0||_{L^2} \leq 1$, we have $||u||_{L^2}^{\frac{10}{3}} \leq ||u||_{L^2}^2$ and

$$\int_0^T\int_0^\infty |u|^3 dxdt \leq \varepsilon \int_0^T\int_0^\infty u_x^2 dxdt + c_\varepsilon \int_0^T\int_0^\infty u^2 dxdt. \tag{49}$$

Moreover, according to [20], there exists $c_1 > 0$, satisfying

$$\int_0^T\int_0^\infty u^2 dxdt \leq c_1\{\frac{1}{2}\int_0^T u_x^2(0,t)dt + \int_0^T\int_0^\infty a(x)u^2 dxdt\}. \tag{50}$$

Now, combining (48)-(50) and taking $\varepsilon < \frac{1}{2}$ and $d_0 := 2c_1(\frac{1}{2} + \frac{c_\varepsilon}{3})$ we obtain

$$\begin{aligned}&V(u(T)) - V(u_0) + \frac{d_0+1}{2}(\frac{1}{2}\int_0^T u_x^2(0,t)dt + \int_0^T\int_0^\infty a(x)|u|^2 dxdt)\\ &+(\frac{3}{2} - \frac{\varepsilon}{3})\int_0^T\int_0^\infty u_x^2 dxdt + \int_0^T\int_0^\infty xa(x)|u|^2 dxdt \leq 0\end{aligned} \tag{51}$$

or

$$V(u(T)) - V(u_0) \leq -\widetilde{c}\{\int_0^T u_x^2(0,t)dt + \int_0^T\int_0^\infty (x+1)a(x)|u|^2 dxdt + \int_0^T\int_0^\infty u_x^2 dxdt\} \tag{52}$$

where $\widetilde{c} > 0$. We aim to prove the existence of a constant $c > 0$ satisfying

$$V(u(T)) - V(u_0) \leq -cV(u_0) \tag{53}$$

Indeed, such an inequality gives at once the decay $V(u(t)) \le ce^{-\nu t}V(u_0)$. To this end, we need to establish two claims.

CLAIM 4. There exists $c > 0$ such that

$$\int_0^T V(u)dt \le c\{\int_0^T u_x^2(0,t)dt + \int_0^T \int_0^\infty (x+1)a(x)u^2dxdt\}.$$

Since $u_0 \in L^2_{(x+1)dx} \subset L^2$, from (4) and (50) we get

$$\begin{aligned}
\int_0^T V(u)dt &= \frac{1}{2}\int_0^T \int_0^\infty (x+1)u^2dxdt + \frac{d_0}{2}\int_0^T \int_0^\infty u^2dxdt \\
&\le \frac{c_1d_0}{2}\{\frac{1}{2}\int_0^T u_x^2(0,t)dt + \int_0^T \int_0^\infty a(x)u^2dxdt\} \\
&\quad + \frac{1}{2}\int_0^T \int_0^{x_0} (x+1)u^2dxdt + \frac{1}{2}\int_0^T \int_{x_0}^\infty (x+1)u^2dxdt \\
&\le \frac{c_1d_0}{2}\{\frac{1}{2}\int_0^T u_x^2(0,t)dt + \int_0^T \int_0^\infty a(x)u^2dxdt\} \\
&\quad + \frac{1}{2}(x_0+1)\int_0^T \int_0^{x_0} u^2dxdt + \frac{1}{2}\int_0^T \int_{x_0}^\infty (x+1)\frac{a(x)}{a_0}u^2dxdt \\
&\le c\{\int_0^T u_x^2(0,t)dt + \int_0^T \int_0^\infty (x+1)a(x)u^2dxdt\}.
\end{aligned}$$

CLAIM 5.

$$V(u_0) \le C(\int_0^T u_x^2(0,t)dt + \int_0^T \int_0^\infty (x+1)a(x)u^2dxdt + \int_0^T \int_0^\infty u_x^2dxdt) \tag{54}$$

where $C > 0$.

Multiplying the first equation in (1) by $(T-t)u$ and integrating by parts in $(0,\infty)\times(0,T)$, we obtain

$$\begin{aligned}
&\frac{T}{2}\int_0^\infty |u_0(x)|^2dx = \\
&\frac{1}{2}\int_0^T \int_0^\infty |u|^2dxdt + \int_0^T \int_0^\infty (T-t)a(x)|u|^2dxdt + \frac{1}{2}\int_0^T (T-t)u_x^2(0,t)dt,
\end{aligned} \tag{55}$$

and therefore, using (50)

$$\int_0^\infty |u_0(x)|^2dx \le C\left(\int_0^T \int_0^\infty a(x)|u|^2dxdt + \int_0^T u_x^2(0,t)dt\right). \tag{56}$$

Now, multiplying by $(T-t)xu$, it follows that

$$\begin{aligned}
&-\frac{T}{2}\int_0^\infty x|u_0(x)|^2dx + \frac{1}{2}\int_0^T \int_0^\infty x|u|^2dxdt + \frac{3}{2}\int_0^T \int_0^\infty (T-t)u_x^2dxdt \\
&-\frac{1}{2}\int_0^T \int_0^\infty (T-t)u^2dxdt + \int_0^T \int_0^\infty (T-t)xa(x)|u|^2dxdt- \\
&\qquad\qquad -\frac{1}{3}\int_0^T \int_0^\infty (T-t)u^3dxdt = 0.
\end{aligned}$$

The identity above and (49) allow us to conclude that

$$\int_0^\infty x|u_0(x)|^2dx$$
$$\leq C\{\int_0^T\int_0^\infty (x+1)|u|^2dxdt+\int_0^T\int_0^\infty u_x^2dxdt+\int_0^T\int_0^\infty xa(x)|u|^2dxdt+ \tag{57}$$
$$+\int_0^T\int_0^\infty |u|^3dxdt\}\leq C\{\int_0^T V(u(t))dt+\int_0^T\int_0^\infty xa(x)u^2dxdt+\int_0^T\int_0^\infty u_x^2dxdt\}$$

for some $C>0$. Claim 5 follows from Claim 4 and (56)-(57). ■

The previous computations give us (53) (and the exponential decay) when $||u_0||_{L^2}\leq 1$. The general case is proved as follows. Let $u_0\in L^2_{(x+1)dx}\subset L^2$ be such that $||u_0||_{L^2}\leq R$. Since $u\in C(\mathbb{R}^+;L^2(\mathbb{R}^+))$ and $||u(t)||_{L^2}\leq \alpha e^{-\beta t}||u_0||_{L^2}$, where $\alpha=\alpha(R)$ and $\beta=\beta(R)$ are positive constants, $||u(T)||_{L^2}\leq 1$ if we pick T satisfying $\alpha e^{-\beta T}R<1$. Then, it follows from (48)-(26) and (53) that for some constants $\nu>0,\ c>0,\ C>0$

$$V(u(t+T))\leq ce^{-\nu t}V(u(T))\leq c(T||u_0||^2_{L^2}+T||u_0||^{\frac{10}{3}}_{L^2}+V(u_0))e^{-\nu t},$$

hence

$$V(u(t))\leq Ce^{-\nu t}V(u_0),$$

where $C=C(R)$, which concludes the proof when $m=1$.

Induction Hypothesis: There exist $c>0$ and $\rho>0$ such that if $V_{m-1}(u_0)\leq\rho$, we have

$$V_m(u)-V_m(u_0) \qquad (*)_m$$
$$\leq -c\{\int_0^T u_x^2(0,t)dt+\int_0^T\int_0^\infty (x+1)^{m-1}u_x^2dxdt+\int_0^T\int_0^\infty (x+1)^m a(x)u^2dxdt\}$$
$$V_m(u_0) \qquad (**)_m$$
$$\leq c\{\int_0^T u_x^2(0,t)dt+\int_0^T\int_0^\infty (x+1)^{m-1}u_x^2dxdt+\int_0^T\int_0^\infty (x+1)^m a(x)u^2dxdt\}.$$

By (52)-(54), the induction hypothesis is true for $m=1$. Pick now an index $m\geq 2$ and assume that $d_0,...,d_{m-2}$ have been constructed so that $(*)_k-(**)_k$ are fulfilled for $1\leq k\leq m-1$. We aim to prove that for a convenient choice of the constant d_{m-1} in (45), the properties $(*)_m-(**)_m$ hold true.

Let us investigate first $(*)_m$. We multiply the first equation in (1) by $(x+1)^m u$ to obtain

$$V_m(u)-V_m(u_0)-d_{m-1}(V_{m-1}(u)-V_{m-1}(u_0))$$
$$-\frac{m(m-1)(m-2)}{2}\int_0^T\int_0^\infty (x+1)^{m-3}u^2dxdt+\frac{1}{2}\int_0^T u_x^2(0,t)dt$$
$$+\frac{3m}{2}\int_0^T\int_0^\infty (x+1)^{m-1}u_x^2dxdt-\frac{m}{2}\int_0^T\int_0^\infty (x+1)^{m-1}u^2dxdt \tag{58}$$
$$-\frac{m}{3}\int_0^T\int_0^\infty (x+1)^{m-1}u^3dxdt+\int_0^T\int_0^\infty (x+1)^m a(x)u^2dxdt=0.$$

The next steps are devoted to estimate the terms in the above identity. First, combining (4) and (50) we infer the existence of a positive constant $c > 0$ such that

$$
\begin{aligned}
&\int_0^T \int_0^\infty (x+1)^{m-1} u^2 dxdt \\
&= \int_0^T \int_0^{x_0} (x+1)^{m-1} u^2 dxdt + \int_0^T \int_{x_0}^\infty (x+1)^{m-1} u^2 dxdt \\
&\leq (x_0+1)^{m-1} \int_0^T \int_0^\infty u^2 dxdt + \frac{1}{a_0} \int_0^T \int_0^\infty a(x)(x+1)^{m-1} u^2 dxdt \\
&\leq c \{ \int_0^T u_x^2(0,t) dt + \int_0^T \int_0^\infty (x+1)^{m-1} a(x) u^2 dxdt \} \\
&\leq -c \{ V_{m-1}(u) - V_{m-1}(u_0) \}
\end{aligned} \tag{59}
$$

where we used $(*)_{m-1}$. In the same way

$$
\begin{aligned}
&\int_0^T \int_0^\infty (x+1)^{m-3} u^2 dxdt \\
&\leq \int_0^T \int_0^\infty (x+1)^{m-1} u^2 dxdt \leq -c \{ V_{m-1}(u) - V_{m-1}(u_0) \}
\end{aligned} \tag{60}
$$

where $c > 0$ is a positive constant. Moreover, assuming $V_{m-1}(u_0) \leq \rho$ with $\rho > 0$ small enough (so that by exponential decay of $V_{m-1}(u(t))$ we have $\int_0^\infty (x+1)^{m-1} |u(x,t)|^2 dx \leq 1$ for all $t \geq 0$) and proceeding as in the case $m = 1$, we obtain the existence of $\varepsilon > 0$ and $c_\varepsilon > 0$ satisfying

$$
\begin{aligned}
&\int_0^T \int_0^\infty (x+1)^{m-1} |u|^3 dxdt \\
&\leq \varepsilon \int_0^T \int_0^\infty (x+1)^{m-1} u_x^2 dxdt + c_\varepsilon \int_0^T \int_0^\infty (x+1)^{m-1} u^2 dxdt.
\end{aligned} \tag{61}
$$

Indeed,

$$
\begin{aligned}
&\int_0^\infty (x+1)^{m-1} |u|^3 dx \\
&\leq ||u||_{L^\infty} \int_0^\infty (x+1)^{m-1} u^2 dx \leq \sqrt{2} ||u_x||_{L^2}^{\frac{1}{2}} ||u||_{L^2}^{\frac{1}{2}} \int_0^\infty (x+1)^{m-1} u^2 dx \\
&\leq \varepsilon \int_0^\infty (x+1)^{m-1} u_x^2 dx + c_\varepsilon \int_0^\infty u^2 dx + c_\varepsilon \left(\int_0^\infty (x+1)^{m-1} u^2 dx \right)^2.
\end{aligned} \tag{62}
$$

Then, if we return to (58) and take $\varepsilon < 9/2$ and $d_{m-1} > 0$ large enough, from (59)-(61) if follows that

$$
\begin{aligned}
&V_m(u) - V_m(u_0) \\
&\leq -c \{ \int_0^T u_x^2(0,t) dt + \int_0^T \int_0^\infty (x+1)^{m-1} u_x^2 dxdt + \int_0^T \int_0^\infty a(x)(x+1)^m u^2 dxdt \} \\
&+ \frac{d_{m-1}}{2} (V_{m-1}(u) - V_{m-1}(u_0)).
\end{aligned} \tag{63}
$$

This yields $(*)_m$, by $(*)_{m-1}$. Let us now check $(**)_m$. It remains to estimate the terms in the right hand side of (63). We multiply the first equation in (1) by $(T-t)(x+1)^m u$ to obtain

$$
\begin{aligned}
&\frac{T}{2}\int_0^\infty (x+1)^m u_0^2 dx = \frac{1}{2}\int_0^T\int_0^\infty (x+1)^m u^2 dxdt \\
&-\frac{m(m-1)(m-2)}{2}\int_0^T\int_0^\infty (T-t)(x+1)^{m-3}u^2 dxdt + \frac{1}{2}\int_0^T (T-t)u_x^2(0,t)dt \\
&+\frac{3m}{2}\int_0^T\int_0^\infty (T-t)(x+1)^{m-1}u_x^2 dxdt - \frac{m}{2}\int_0^T\int_0^\infty (T-t)(x+1)^{m-1}u^2 dxdt \\
&-\frac{m}{3}\int_0^T\int_0^\infty (T-t)(x+1)^{m-1}u^3 dxdt + \int_0^T\int_0^\infty (T-t)(x+1)^m a(x)u^2 dxdt.
\end{aligned}
$$

Then, proceeding as above, we deduce that

$$
\begin{aligned}
&\int_0^T (x+1)^m u_0^2 dx \\
&\le c\{\int_0^T\int_0^\infty (x+1)^{m-1}u^2 dxdt + \int_0^T u_x^2(0,t)dt + \int_0^T\int_0^\infty (x+1)^{m-1}u_x^2 dxdt \\
&+\int_0^T\int_0^\infty (x+1)^m a(x)u^2 dxdt\} \\
&\le c\{\int_0^T u_x^2(0,t)dt + \int_0^T\int_0^\infty (x+1)^{m-1}u_x^2 dxdt + \int_0^T\int_0^\infty (x+1)^m a(x)u^2 dxdt\}.
\end{aligned}
$$

Combined to $(**)_{m-1}$, this yields $(**)_m$. This completes the construction of the sequence $\{V_m\}_{m\ge1}$ by induction.

Let us now check the exponential decay of V_m for $m \ge 2$. It follows from $(*)_m - (**)_m$ that

$$V_m(u) - V_m(u_0) \le -c\, V_m(u_0)$$

where $c > 0$, which completes the proof when $V_{m-1}(u_0) \le \rho$. The global result $(V_{m-1}(u_0) \le R)$ is obtained as above for $m = 1$. ■

Corollary 3.2 *Let $a = a(x)$ fulfilling (4) and $a \in W^{2,\infty}(0,\infty)$. Then for any $R > 0$, there exist positive constants $c = c(R)$ and $\mu = \mu(R)$ such that*

$$||u_x(t)||_{L^2(\mathbb{R}^+)} \le c\frac{e^{-\mu t}}{\sqrt{t}}||u_0||_{L^2_{(x+1)dx}} \tag{64}$$

for all $t > 0$ and all $u_0 \in L^2_{(x+1)dx}$ satisfying $||u_0||_{L^2_{(x+1)dx}} \le R$.

Proof. Pick any $R > 0$ and any $u_0 \in L^2_{(x+1)dx}$ with $||u_0||_{L^2_{(x+1)dx}} \le R$. By Theorem 3.1 there are some constants $C = C(R)$ and $\nu = \nu(R)$ such that

$$||u(t)||_{L^2_{(x+1)dx}} \le Ce^{-\nu t}||u_0||_{L^2_{(x+1)dx}}. \tag{65}$$

Using the multiplier $t(u^2 + 2u_{xx})$ we obtain after some integrations by parts that for all $0 < t_1 < t_2$

$$
\begin{aligned}
&t_2\int_0^\infty u_x^2(x,t_2)dx + \int_{t_1}^{t_2} tu_x^2(0,t)dt + 2\int_{t_1}^{t_2}\int_0^\infty ta(x)u_x^2 dxdt + \int_{t_1}^{t_2} tu_{xx}^2(0,t)dt \\
&= -\frac{1}{3}\int_{t_1}^{t_2}\int_0^\infty u^3 dxdt + \frac{t_2}{3}\int_0^\infty u^3(x,t_2)dx + \int_{t_1}^{t_2}\int_0^\infty tu^3 a(x)dxdt \\
&+\int_{t_1}^{t_2}\int_0^\infty u_x^2 dxdt + \int_{t_1}^{t_2}\int_0^\infty ta''(x)u^2 dxdt.
\end{aligned} \tag{66}
$$

1. Let us assume first that $T > 1$. Applying (66) on the time interval $[T-1,T]$, we infer that

$$\int_0^\infty |u_x(x,T)|^2 dx \le c\left(\int_{T-1}^T\int_0^\infty |u|^3 dxdt + ||u(T)||^3_{L^3(\mathbb{R}^+)} + \int_{T-1}^T ||u||^2_{H^1(\mathbb{R}^+)} dt\right). \tag{67}$$

To estimate the cubic terms in (67), we use (26) to obtain

$$\begin{aligned}\int_0^\infty |u_x(x,T)|^2 dx &\le \varepsilon \int_0^\infty |u_x(x,T)|^2 dx \\ &+ c_\varepsilon\big(||u(T)||^{\frac{10}{3}}_{L^2(\mathbb{R}^+)} + \int_{T-1}^T (||u||^2_{H^1(\mathbb{R}^+)} + ||u||^{\frac{10}{3}}_{L^2(\mathbb{R}^+)})dt\big).\end{aligned} \tag{68}$$

Note that by (65)

$$||u(T)||^{\frac{10}{3}}_{L^2(\mathbb{R}^+)} \le (Ce^{-\nu T}||u_0||_{L^2_{(x+1)dx}})^{\frac{10}{3}} \le C^{\frac{10}{3}} R^{\frac{4}{3}} e^{-\nu T}||u_0||^2_{L^2_{(x+1)dx}}.$$

It follows from (48), (26), and (65) that

$$\begin{aligned}&\int_{T-1}^T (||u||^2_{H^1(\mathbb{R}^+)} + ||u||^{\frac{10}{3}}_{L^2(\mathbb{R}^+)})dt \\ &\quad \le C\left(V_1(u(T-1)) + \int_{T-1}^T \big(||u||^2_{L^2(\mathbb{R}^+)} + ||u||^{\frac{10}{3}}_{L^2(\mathbb{R}^+)}\big)dt\right) \\ &\quad \le Ce^{-\nu T}||u_0||^2_{L^2_{(x+1)dx}}\end{aligned} \tag{69}$$

where $C = C(R,\nu)$. (64) for $T \ge 1$ follows from (68) and (69) by choosing $\varepsilon < 1$ and $\mu < \nu$.
2. Assume now that $T \le 1$. Estimating again the cubic terms in (66) (with $[t_1,t_2] = [0,T]$) by using (26), we obtain

$$\begin{aligned}T\int_0^\infty u_x^2(x,T)dx &\le \frac{T}{3}\left(\varepsilon||u_x(T)||^2_{L^2(\mathbb{R}^+)} + C_\varepsilon||u(T)||^{\frac{10}{3}}_{L^2(\mathbb{R}^+)}\right) \\ &\quad + C_\varepsilon\int_0^T (||u||^2_{H^1(\mathbb{R}^+)} + ||u||^{\frac{10}{3}}_{L^2(\mathbb{R}^+)})dt.\end{aligned} \tag{70}$$

By (48), (26) and (65), we have that

$$\int_0^1\int_0^\infty |u_x|^2 dxdt \le C(R)||u_0||^2_{L^2_{(x+1)dx}} \tag{71}$$

which, combined to (70) with $\varepsilon = 1$ and (65), gives

$$||u_x(T)||^2_{L^2(\mathbb{R}^+)} \le C(R)T^{-1}||u_0||^2_{L^2_{(x+1)dx}}$$

for all $T < 1$. This gives (64) for $T < 1$. ■
Corollary 3.2 may be extended (locally) to the weighted space $L^2_{(x+1)^m dx}$ ($m \ge 2$) in following the method of proof of [24, Theorem 1.1].

Corollary 3.3 *Let $a = a(x)$ fulfilling (4) and $m \ge 2$. Then there exist some constants $\rho > 0$, $C > 0$ and $\mu > 0$ such that*

$$||u(t)||_{H^1_{(x+1)^m dx}} \le C\frac{e^{-\mu t}}{\sqrt{t}}||u_0||_{L^2_{(x+1)^m dx}}$$

for all $t > 0$ and all $u_0 \in L^2_{(x+1)^m dx}$ satisfying $||u_0||_{L^2_{(x+1)^m dx}} \le \rho$.

Proof. We first prove estimates for the linearized problem

$$u_t + u_x + u_{xxx} + au = 0 \tag{72}$$

$$u(0,t) = 0 \tag{73}$$

$$u(x,0) = u_0(x) \tag{74}$$

and next apply a perturbation argument to extend them to the nonlinear problem (1). Let us denote by $W(t)u_0 = u(t)$ the solution of (72)-(74). By computations similar to those performed in the proof of Theorem 3.1, we have that

$$||W(t)u_0||_{L^2_{(x+1)^m dx}} \le C_0 e^{-\nu t} ||u_0||_{L^2_{(x+1)^m dx}}.$$

We need the
CLAIM 6. Let $k \in \{0, ..., 3\}$. Then there exists a constant $C_k > 0$ such that for any $u_0 \in H^k_{(x+1)^m dx}$,

$$||W(t)u_0||_{H^k_{(x+1)^m dx}} \le C_k e^{-\nu t} ||u_0||_{H^k_{(x+1)^m dx}}. \tag{75}$$

Indeed, if $u_0 \in H^3_{(x+1)^m dx}$, then $u_t(.,0) \in L^2_{(x+1)^{m-3} dx}$, and since $v = u_t$ solves (72)-(73), we also have that

$$||u_t(.,t)||_{L^2_{(x+1)^{m-3} dx}} \le C_0 e^{-\nu t} ||u_t(.,0)||_{L^2_{(x+1)^{m-3} dx}}.$$

Using (72), this gives

$$||W(t)u_0||_{H^3_{(x+1)^m dx}} \le C_3 e^{-\nu t} ||u_0||_{H^3_{(x+1)^m dx}}.$$

This proves (75) for $k = 3$. The fact that (75) is valid for $k = 1, 2$ follows from a standard interpolation argument, for $H^k_{(x+1)^m dx} = [H^0_{(x+1)^m dx}, H^3_{(x+1)^m dx}]_{\frac{k}{3}}$.

Lemma 3.4 *Pick any number $\mu \in (0, \nu)$. Then there exists some constant $C = C(\mu) > 0$ such that for any $u_0 \in L^2_{(x+1)^m dx}$*

$$||W(t)u_0||_{H^1_{(x+1)^m dx}} \le C \frac{e^{-\mu t}}{\sqrt{t}} ||u_0||_{L^2_{(x+1)^m dx}}. \tag{76}$$

Proof. Let $u_0 \in L^2_{(x+1)^m dx}$ and set $u(t) = W(t)u_0$ for all $t \ge 0$. By scaling in (72) by $(x+1)^m u$, we see that for some constant $C_K = C_K(T)$

$$||u||_{L^2(0,1;H^1_{(x+1)^m dx})} \le C_K ||u_0||_{L^2_{(x+1)^m dx}}.$$

This implies that $u(t) \in H^1_{(x+1)^m dx}$ for a.e. $t \in (0,1)$ which, combined to (75), gives that $u(t) \in H^1_{(x+1)^m dx}$ for all $t > 0$. Pick any $T \in (0, 1]$. Note that, by (75),

$$||u(T)||_{H^1_{(x+1)^m dx}} \le C_1 e^{-\nu(T-t)} ||u(t)||_{H^1_{(x+1)^m dx}}, \quad \forall t \in (0,T). \tag{77}$$

Integrating with respect to t in (77) yields

$$[C_1^{-1} ||u(T)||_{H^1_{(x+1)^m dx}}]^2 \int_0^T e^{2\nu(T-t)} dt \le \int_0^T ||u(t)||^2_{H^1_{(x+1)^m dx}} dt,$$

and hence

$$\begin{aligned} ||u(T)||_{H^1_{(x+1)^m dx}} &\le C_K C_1 \sqrt{\frac{2\nu}{e^{2\nu T} - 1}} ||u_0||_{L^2_{(x+1)^m dx}} \\ &\le \frac{C_K C_1}{\sqrt{T}} ||u_0||_{L^2_{(x+1)^m dx}} \end{aligned}$$

for $0 < T \leq 1$. Therefore

$$||u(t)||_{H^1_{(x+1)^m dx}} \leq C_K\, C_1 e^{\nu} \frac{e^{-\nu t}}{\sqrt{t}} ||u_0||_{L^2_{(x+1)^m dx}} \qquad \forall t \in (0,1). \tag{78}$$

(76) follows from (78) and (75), since $\mu < \nu$. ■

Let us return to the proof of Corollary 3.3. Fix a number $\mu \in (0, \nu)$, where ν is as in (75), and let us introduce the space

$$F = \{u \in C(\mathbb{R}^+; H^1_{(x+1)^m dx}); \quad ||e^{\mu t} u(t)||_{L^\infty(\mathbb{R}^+; H^1_{(x+1)^m dx})} < \infty\}$$

endowed with its natural norm. Note that (1) may be recast in the following integral form

$$u(t) = W(t)u_0 + \int_0^t W(t-s)N(u(s))\, ds \tag{79}$$

where $N(u) = -uu_x$. We first show that (79) has a solution in F provided that $u_0 \in H^1_{(x+1)^m dx}$ with $||u_0||_{H^1_{(x+1)^m dx}}$ small enough. Let $u_0 \in H^1_{(x+1)^m dx}$ and $u \in F$ with $||u_0||_{H^1_{(x+1)^m dx}} \leq r_0$ and $||u||_F \leq R$, r_0 and R being chosen later. We introduce the map Γ defined by

$$(\Gamma u)(t) = W(t)u_0 + \int_0^t W(t-s)N(u(s))\, ds \qquad \forall t \geq 0. \tag{80}$$

We shall prove that Γ has a fixed point in the closed ball $B_R(0) \subset F$ provided that $r_0 > 0$ is small enough.

For the forcing problem

$$\begin{cases} u_t + u_x + u_{xxx} + au = f \\ u(0,t) = 0 \\ u(x,0) = u_0(x) \end{cases}$$

we have the following estimate

$$\begin{aligned} &\sup_{0\leq t\leq T} ||u(t)||^2_{L^2_{(x+1)^m dx}} + \int_0^T\int_0^\infty (x+1)^{m-1} u_x^2 dxdt \\ &\leq C\left(||u_0||^2_{L^2_{(x+1)^m dx}} + ||f||^2_{L^1(0,T;L^2_{(x+1)^m dx})}\right). \end{aligned}$$

Let us take $f = N(u) = -uu_x$. Observe that for all $x > 0$

$$\begin{aligned} (x+1)u^2(x) &= \left| \int_0^\infty \frac{d}{dx}[(x+1)u^2(x)]dx \right| \\ &\leq C\left(\int_0^\infty (x+1)^m |u|^2 dx + \int_0^\infty (x+1)^{m-1}|u_x|^2 dx \right) \end{aligned}$$

whenever $m \geq 2$. It follows that for some constant $K > 0$

$$\begin{aligned} ||uu_x||^2_{L^2_{(x+1)^m dx}} &\leq ||(x+1)u^2||_{L^\infty(\mathbb{R}^+)} \int_0^\infty (x+1)^{m-1}|u_x|^2 dx \\ &\leq K||u||^4_{H^1_{(x+1)^m dx}}. \end{aligned}$$

Therefore, for any $T > 0$,

$$\begin{aligned} &\sup_{0\leq t\leq T} ||(\Gamma u)(t)||^2_{L^2_{(x+1)^m dx}} + \int_0^T\int_0^\infty (x+1)^{m-1}|(\Gamma u)_x|^2 dxdt \\ &\leq C\left(||u_0||^2_{L^2_{(x+1)^m dx}} + \Big(\int_0^T ||u(t)||^2_{H^1_{(x+1)^m dx}} dt\Big)^2\right) < \infty. \end{aligned}$$

Thus $\Gamma u \in C(\mathbb{R}^+, L^2_{(x+1)^m dx}) \cap L^2_{loc}(\mathbb{R}^+; H^1_{(x+1)^m dx})$ with $(\Gamma u)(0) = u_0$. We claim that $\Gamma u \in F$. Indeed, by (75),

$$||e^{\mu t} W(t) u_0||_{H^1_{(x+1)^m dx}} \le C_1 ||u_0||_{H^1_{(x+1)^m dx}}$$

and for all $t \ge 0$

$$\begin{aligned}
||e^{\mu t} \int_0^t W(t-s) N(u(s)) ds||_{H^1_{(x+1)^m dx}} &\le Ce^{\mu t} \int_0^t \frac{e^{-\mu(t-s)}}{\sqrt{t-s}} ||N(u(s))||_{L^2_{(x+1)^m dx}} ds \\
&\le C \int_0^t \frac{e^{\mu s}}{\sqrt{t-s}} K(e^{-\mu s} ||u||_F)^2 ds \\
&\le CK ||u||_F^2 \int_0^t \frac{e^{-\mu(t-s)}}{\sqrt{s}} ds \\
&\le CK(2 + \mu^{-1}) ||u||_F^2
\end{aligned}$$

where we used Lemma 3.4. Pick $R > 0$ such that $CK(2+\mu^{-1})R \le \frac{1}{2}$, and r_0 such that $C_1 r_0 = \frac{R}{2}$. Then, for $||u_0||_{H^1_{(x+1)^m dx}} \le r_0$ and $||u||_F \le R$, we obtain that

$$||e^{\mu t} (\Gamma u)(t)||_{H^1_{(x+1)^m dx}} \le C_1 r_0 + CK(2+\mu^{-1}) R^2 \le R, \quad t \ge 0.$$

Hence Γ maps the ball $B_R(0) \subset F$ into itself. Similar computations show that Γ contracts. By the contraction mapping theorem, Γ has a unique fixed point u in $B_R(0)$. Thus $||u(t)||_{H^1_{(x+1)^m dx}} \le Ce^{-\mu t} ||u_0||_{H^1_{(x+1)^m dx}}$ provided that $||u_0||_{H^1_{(x+1)^m dx}} \le r_0$ with r_0 small enough. Proceeding as in the proof of Lemma 3.4, we have that

$$||u(t)||_{H^1_{(x+1)^m dx}} \le C \frac{e^{-\mu t}}{\sqrt{t}} ||u_0||_{L^2_{(x+1)^m dx}} \qquad \text{for } 0 < t < 1,$$

provided that $||u_0||_{L^2_{(x+1)^m dx}} \le \rho_0$ with $\rho_0 < 1$ small enough. The proof is complete with a decay rate $\mu' < \mu$. ■

Corollary 3.5 *Assume that $a(x)$ satisfies (4) and that $\partial_x^k a \in L^\infty(\mathbb{R}^+)$ for all $k \ge 0$. Pick any $u_0 \in L^2_{(x+1)^m dx}$. Then for all $\varepsilon > 0$, all $T > \varepsilon$, and all $k \in \{1, ..., m\}$, there exists a constant $C = C(\varepsilon, T, k) > 0$ such that*

$$\int_\varepsilon^\infty (x+1)^{m-k} |\partial_x^k u(x,t)|^2 dx \le C ||u_0||^2_{L^2_{(x+1)^m dx}} \qquad \forall t \in [\varepsilon, T]. \tag{81}$$

Proof. The proof is very similar to the one in [18, Lemma 5.1] and so we only point out the small changes. First, it should be noticed that the presence in the KdV equation of the extra terms u_x and $a(x)u$ does not cause any serious trouble. On the other hand, choosing a cut-off function in x of the form $\eta(x) = \psi_0(x/\varepsilon)$ (instead of $\eta(x) = \psi_0(x - x_0 + 2)$ as in [18]) where $\psi_0 \in C^\infty(\mathbb{R}, [0,1])$ satisfies $\psi_0(x) = 0$ for $x \le 1/2$ and $\psi_0(x) = 1$ for $x \ge 1$, allows to overcome the fact that u is a solution of (1) on the half-line only. ■

3.2 Decay in L^2_b

This section is devoted to the exponential decay in L^2_b. Our result reads as follows:

Theorem 3.6 *Assume that the function $a = a(x)$ satisfies (4) with $4b^3 + b < a_0$. Then, for all $R > 0$, there exist $C > 0$ and $\nu > 0$, such that*

$$||u(t)||_{L^2_b} \leq Ce^{-\nu t}||u_0||_{L^2_b} \qquad t \geq 0$$

for any solution u given by Theorem 2.3.

Proof. We introduce the Lyapunov function

$$V(u) = \frac{1}{2}\int_0^\infty u^2 e^{2bx}dx + c_b \int_0^\infty u^2 dx, \tag{82}$$

where c_b is a positive constant that will be chosen later. Then, adding (17) and (18) hand by hand we obtain

$$\begin{aligned} V(u) - V(u_0) = {} & (4b^3 + b)\int_0^T \int_{x_0}^\infty u^2 e^{2bx}dxdt + (4b^3 + b)\int_0^T \int_0^{x_0} u^2 e^{2bx}dxdt \\ & - 3b\int_0^\infty \int_0^\infty u_x^2 e^{2bx}dxdt + \frac{2b}{3}\int_0^T \int_0^\infty u^3 e^{2bx}dxdt \\ & - (c_b + \frac{1}{2})\int_0^T u_x^2(0,t)dt - \int_0^T \int_0^\infty a(x)|u|^2(e^{2bx} + 2c_b)dxdt, \end{aligned} \tag{83}$$

where x_0 is the number introduced in (4). On the other hand, since $L^2_b \subset L^2_{(x+1)dx}$, $||u(t)||_{L^2(0,\infty)}$ and $||u_x(t)||_{L^2(0,\infty)}$ decays to zero exponentially. Consequently, from Moser estimate we deduce that $||u(t)||_{L^\infty(0,\infty)} \to 0$. We may assume that $(2b/3)||u(t)||_{L^\infty} < \varepsilon = a_0 - (4b^3 + b)$ for all $t \geq 0$, by changing u_0 into $u(t_0)$ for t_0 large enough. Therefore

$$\begin{aligned} & \frac{2b}{3}\int_0^T \int_0^\infty |u|^3 e^{2bx}dxdt \\ & \leq \frac{2b}{3}\int_0^T ||u(t)||_{L^\infty(0,\infty)}\left(\int_0^\infty |u|^2 e^{2bx}dx\right)dt \leq \varepsilon\int_0^T \int_0^\infty u^2 e^{2bx}dxdt. \end{aligned} \tag{84}$$

So, returning to (83), the following holds

$$\begin{aligned} & V(u) - V(u_0) - (4b^3 + b + \varepsilon)\int_0^T \int_0^{x_0} u^2 e^{2bx}dxdt \\ & +3b\int_0^T \int_0^\infty u_x^2 e^{2bx}dxdt + (c_b + \frac{1}{2})\int_0^T u_x^2(0,t)dt + 2c_b\int_0^T \int_0^\infty a(x)|u|^2dxdt \leq 0. \end{aligned} \tag{85}$$

Moreover, according to [20] there exists $C > 0$ satisfying

$$\begin{aligned} & \int_0^T \int_0^{x_0} u^2 e^{2bx}dxdt \\ & \leq e^{2bx_0}\int_0^T \int_0^{x_0} u^2 dxdt \leq C\{\int_0^T u_x^2(0,t)dt + \int_0^T \int_0^\infty a(x)u^2 dxdt\} \end{aligned}$$

since $L^2_b \subset L^2(\mathbb{R}^+)$. Then, choosing c_b sufficiently large, the above estimate and (85) give us that

$$\begin{aligned} V(u) - V(u_0) \leq {} & -C\{\int_0^T u_x^2(0,t)dt + \int_0^T \int_0^\infty a(x)u^2 dxdt \\ & + \int_0^T \int_0^\infty u_x^2 e^{2bx}dxdt\} \leq -C\,V(u_0), \end{aligned} \tag{86}$$

which allows to conclude that $V(u)$ decays exponentially. The last inequality is a consequence of the following results:

Claim 7. There exists a positive constant $C>0$, such that

$$\int_0^T V(u(t))dt \leq C\int_0^T\int_0^\infty u_x^2 e^{2bx}dxdt.$$

First, observe that

$$|\int_0^\infty u^2e^{2bx}dx| = |-\frac{1}{b}\int_0^\infty uu_xe^{2bx}dx| \leq \frac{1}{b}(\int_0^\infty u^2e^{2bx}dx)^{\frac{1}{2}}(\int_0^\infty u_x^2e^{2bx}dx)^{\frac{1}{2}},$$

therefore,

$$\int_0^\infty u^2e^{2bx}dx \leq \frac{1}{b^2}\int_0^\infty u_x^2e^{2bx}dx. \tag{87}$$

Then, from (4) and (87) we have

$$V(u(t)) \leq (\frac{1}{2}+c_b)\int_0^\infty u^2e^{2bx}dx \leq (\frac{1}{2}+c_b)b^{-2}\int_0^\infty u_x^2e^{2bx}dx$$

which gives us Claim 7.

Claim 8.

$$V(u_0) \leq C\{\int_0^T u_x^2(0,t)dt + \int_0^T\int_0^\infty u_x^2e^{2bx}dxdt + \int_0^T V(u(t))dt\},$$

where C is a positive constant.

Multiplying the first equation in (1) by $(T-t)ue^{2bx}$ and integrating by parts in $(0,\infty)\times(0,T)$, we obtain

$$\begin{aligned}
&-\frac{T}{2}\int_0^\infty |u_0(x)|^2e^{2bx}dx + \frac{1}{2}\int_0^T\int_0^\infty |u|^2e^{2bx}dxdt + 3b\int_0^T\int_0^\infty (T-t)u_x^2e^{2bx}dxdt\\
&+\frac{1}{2}\int_0^T (T-t)u_x^2(0,t)dt - (4b^3+b)\int_0^T\int_0^\infty (T-t)u^2e^{2bx}dxdt\\
&+\int_0^T\int_0^\infty (T-t)a(x)|u|^2e^{2bx}dxdt - \frac{2b}{3}\int_0^T\int_0^\infty (T-t)u^3e^{2bx}dxdt = 0
\end{aligned} \tag{88}$$

and therefore,

$$\begin{aligned}
&\int_0^\infty |u_0(x)|^2e^{2bx}dx \leq C(\int_0^T u_x^2(0,t)dt + \frac{1}{2}\int_0^T\int_0^\infty u^2e^{2bx}dxdt\\
&+\int_0^T\int_0^\infty u_x^2e^{2bx}dxdt + \int_0^T\int_0^\infty |u|^3e^{2bx}dxdt).
\end{aligned} \tag{89}$$

Then, combining (87) and (84), we derive Claim 8. (86) follows at once. This proves the exponential decay when $||u(t)||_{L^\infty} \leq 3\varepsilon/(2b)$. The general case is obtained as in Theorem 3.1 ■

Corollary 3.7 *Assume that the function $a=a(x)$ satisfies (4) with $4b^3+b<a_0$. Then for any $R>0$, there exist positive constants $c=c(R)$ and $\mu=\mu(R)$ such that*

$$||u_x(t)||_{L_b^2} \leq c\frac{e^{-\mu t}}{\sqrt{t}}||u_0||_{L_b^2} \tag{90}$$

for all $t>0$ and all $u_0\in L_b^2$ satisfying $||u_0||_{L_b^2}\leq R$.

Corollary 3.8 *Assume that the function $a = a(x)$ satisfies (4) with $4b^3 + b < a_0$, and let $s \geq 2$. Then there exist some constants $\rho > 0$, $C > 0$ and $\mu > 0$ such that*

$$||u(t)||_{H_b^s} \leq C \frac{e^{-\mu t}}{t^{\frac{s}{2}}} ||u_0||_{L_b^2}$$

for all $t > 0$ and all $u_0 \in L_b^2$ satisfying $||u_0||_{L_b^2} \leq \rho$.

The proof of Corollary 3.7 (resp. 3.8) is very similar to the proof of Corollary 3.2 (resp. 3.3), so it is omitted.

Acknowledgments.

This work was achieved while the first author (AP) was visiting Université Paris-Sud with the support of the Cooperation Agreement Brazil-France and the second author (LR) was visiting IMPA and UFRJ. LR was partially supported by the "Agence Nationale de la Recherche" (ANR), Project CISIFS, Grant ANR-09-BLAN-0213-02.

References

[1] J. L. Bona and R. Winther, *The Korteweg-de Vries equation,posed in a quarter-plane*, SIAM J. Math. Anal. **14** (1983), 1056–1106.

[2] J. L. Bona, W. G. Pritchard and L. R. Scott, *An evaluation of a model equation for water waves*, Philos. Trans. Royal Soc. London, Series A, **302** (1981), 457–510.

[3] J. L. Bona and P. J. Bryant, *A mathematical model for long waves generated by wavemakers in non-linear dispersive systems*, Proc. Cambridge Philos. Soc., **73** (1973), 391–405.

[4] J. L. Bona, S. M. Sun and B.-Y. Zhang, *A non-homogeneous boundary-value problem for the Korteweg-de Vries equation in a quarter plane*, Trans. American Math. Soc., **354** (2002), 427–490.

[5] J. L. Bona, S. M. Sun and B.-Y. Zhang, *A forced oscillations of a damped Korteweg-de Vries equation in a quarter plane*, Comm. Cont. Math. **5** (2003), 369–400.

[6] J. L. Bona, S. M. Sun and B.-Y. Zhang, *Boundary smoothing properties of the Korteweg-de Vries equation in a quarter plane and applications*, Dynamics Partial Differential Eq. **3** (2006), 1–70.

[7] J. L Bona, S. M. Sun, and B. Y. Zhang, *Nonhomogeneous problems for the Korteweg-de Vries and the Korteweg-de Vries-Burgers equations in a quarter plane*, Ann. Inst. H. Poincaré Anal. Non Linéaire **25** (2008), 1145–1185.

[8] J. L. Bona and Jiahong Wu, *Temporal growth and eventual periodicity for dispersive wave equations in a quarter plane*, Discrete Contin. Dyn. Syst. **23** (2009), 1141–1168.

[9] J. Boussinesq, *Essai sur la théorie des eaux courantes; Mémoires présentés par divers savants*, à l'Acad. des Sci. Inst. Nat. France, **23** (1877), 1C680.

[10] E. Cerpa and E. Crépeau, *Rapid exponential stabilization for a linear Korteweg-de Vries equation*, Discrete Contin. Dyn. Syst. Ser. B, **11** (2009), no. 3, 655–668.

[11] J. E. Colliander and C. E. Kenig, *The generalized Korteweg-de Vries equation on the half line*, Comm. Partial Diff. Eq., **27** (2002), 2187–2266.

[12] A. V. Faminskii, *A mixed problem in a semistrip for the Korteweg-de Vries equation and its generalizations*, (Russian) Dinamika Sploshn Sredy, **51** (1988), 54–94.

[13] A. V. Faminskii, *An initial boundary-value problem in a half- strip for the Korteweg-de Vries equation in fractional-order Sobolev spaces. Comm. Partial Differential Equations* **29** (2004), no. 11-12, 1653–1695.

[14] C. S. Gardner, J. M. Greene, M. D. Kruskal and R. M. Miura, *Method for solving the Korteweg-de Vries equation*, Phys. Rev. Lett., **19** (1967), 1095–1097.

[15] T. Kato, *On the Cauchy problem for the (Generalized) Korteweg-de Vries Equation*, Stud. Appl. Math. Adv. Math. Suppl. Stud. **8** (1983), 93–128.

[16] E. M. de Jager, *On the origin of the Korteweg-de Vries equation*, arXiv:math.HO/0602661.

[17] D. J. Korteweg and G. de Vries, *On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves*, Philos. Mag., **39** (1895), 422–443.

[18] S. N. Kruzhkov, A. V. Faminskii *Generalized solutions of the Cauchy problem for the Korteweg-de Vries equation*, Mat. Sb. **120** (1983) (162)(3):346–425. English transl. in Sb. Math. 1984, **48**(2):391–421.

[19] J. A. Leach and D. J. Needham *The large-time development of the solution to an initial-value problem for the Korteweg-de Vries equation. I. Initial data has a discontinuous expansive step*, Nonlinearity **21** (2008), 2391–2408.

[20] F. Linares and A. F. Pazoto, *Asymptotic behavior of the Korteweg-de Vries equation posed in a quarter plane*, J. Differential Equations **246** (2007), 1342–1353.

[21] J.-L. Lions and E. Magenes, "Problèmes aux limites non homogènes et applications", Tome 1, Dunod, Paris, 1968.

[22] R. M. Miura, *The Korteweg-de Vries equation: A survey of results*, SIAM Rev., **18** (1976), 412–459.

[23] A. Pazoto, *Unique continuation and decay for the Korteweg-de Vries equation with localized damping*, ESAIM Control Optim. Calc. Var. **11** (2005), 473–486.

[24] A. Pazoto and L. Rosier, *Stabilization of a Boussinesq system of KdV-KdV type*, Systems & Control Lett. **57** (2008), 595–601.

[25] G. Perla Menzala, C.F. Vasconcellos and E. Zuazua, *Stabilization of the Korteweg-de Vries equation with localized damping*, Quart. Appl. Math. **60** (2002), 111–129.

[26] L. Rosier, *Exact boundary controllability for the Korteweg-de Vries equation on a bounded domain*, ESAIM Control Optim. Calc. Var. **2** (1997), 33–55 (electronic).

[27] L. Rosier, *Exact boundary controllability for the linear Korteweg-de Vries equation on the half-line*, SIAM J. Control Optim. **39** (2000), 331–351.

[28] L. Rosier, *A fundamental solution supported in a strip for a dispersive equation*, Computational and Applied Mathematics **21** (2002), 355–367.

[29] L. Rosier, *Control of the surface of a fluid by a wavemaker*, ESAIM Control Optim. Calc. Var. **10** (2004), 346–380

[30] L. Rosier and B.-Y. Zhang, *Global stabilization of the generalized Korteweg-de Vries equation posed on a finite domain*, SIAM J. Control Optim. **45** (2006), 927–956.

[31] M. E. Taylor, "Partial Differential Equations III, Nonlinear Equations", Series: Applied Mathematical Sciences 117, Springer-Verlag New York Inc., 1996.